# Newton's Method and Self-Concordance

- Differentiable convex optimization and acceleration
- Newton's Method
- Armijo backtracking search
- self-concordant functions
- Interior Point Method

# Unconstrained Differentiable Convex Optimization

$$\min_x f(x)$$

- $f(x)$ strongly convex and differentiable
- $\partial f(x) = \{\nabla f(x)\}$
- subgradient descent = gradient descent

# Gradient Descent for Strongly Convex Functions

- recall strong convexity

  A convex function $f$ is called strongly convex if there exists two positive constants $\beta_- \leq \beta_+$ such that

$$\beta_- I \preceq \nabla^2 f(x) \preceq \beta_+ I$$

  for every $x$ in the domain of $f$

- Equivalent to

$$\lambda_{\min}(\nabla^2 f(x)) \geq \beta_-$$
$$\lambda_{\max}(\nabla^2 f(x)) \leq \beta_+$$

# Gradient Descent for Strongly Convex Functions

$$x_{t+1} = x_t - \mu_t \nabla f(x_t)$$

- Suppose that $f$ is strongly convex with parameters $\beta_-, \beta_+$

  define $f^* := \min_x f(x)$

  **Convergence result:** Using constant step-size $\mu_t = \frac{1}{\beta_+}$, we have

$$f(x_{t+1}) - f^* \leq (1 - \frac{\beta_-}{\beta_+})(f(x_t) - f^*)$$

  recursively applying we get

- $f(x_k) - f^* \leq (1 - \frac{\beta_-}{\beta_+})^k (f(x_0) - f^*)$

# Gradient Descent for Strongly Convex Functions

- linear convergence

- rate depends on the curvature

$$f(x_k) - f^* \leq (1 - \frac{\beta_-}{\beta_+})^k (f(x_0) - f^*)$$

- minimizing $f(Ax)$ where $A \in \mathbb{R}^{n \times d}$ via Gradient Descent takes

  $O(\kappa nd \log(\frac{1}{\epsilon}))$ operations where $\kappa = \frac{\beta_+}{\beta_-}$

# Gradient Descent with Momentum (Heavy Ball Method) for Strongly Convex Functions

- $x_{t+1} = x_t - \mu \nabla f(x_t) + \beta(x_t - x_{t-1})$
- step-size parameter $\mu = \frac{4}{(\sqrt{\beta_+}+\sqrt{\beta_-})^2}$
- momentum parameter $\beta = \max\left(|1 - \sqrt{\mu\beta_-}|, |1 - \sqrt{\mu\beta_+}|\right)^2$
- minimizing $f(Ax)$ where $A \in \mathbb{R}^{n\times d}$ via Gradient Descent with Momentum takes $O(\sqrt{\kappa}nd\log(\frac{1}{\epsilon}))$ where $\kappa = \frac{\beta_+}{\beta_-}$

# Newton's Method

- Suppose $f$ is twice differentiable, and consider a second order Taylor approximation at a point $x_t$

$$f(y) \approx f(x_t) + \nabla f(x_t)^T (y - x_t) + \frac{1}{2}(y - x^t)\nabla^2 f(x^t)(y - x^t)$$

- minimizing the approximation yields $x_{t+1} = x_t - \left(\nabla^2 f(x)\right)^{-1} \nabla f(x)$

- Damped Newton updates: $x_{t+1} = x_t - t\Delta_t$ where $\Delta_t := \left(\nabla^2 f(x)\right)^{-1} \nabla f(x)$, where $t$ is a step-size parameter

- Hessian of $f(Ax)$ where $A \in \mathbb{R}^{n\times d}$ takes $O(nd^2)$ operations to calculate and $O(d^3)$ to invert. Alternatively, we can factorize in $O(nd^2)$ time (QR, Cholesky, SVD)

# Choosing step-sizes: backtracking (Armijo) line search

**given** a descent direction $\Delta x$ for $f$ at $x \in \mathbf{dom}\, f$, $\alpha \in (0, 0.5)$, $\beta \in (0, 1)$.

$t := 1.$

**while** $f(x + t\Delta x) > f(x) + \alpha t \nabla f(x)^T \Delta x, \quad t := \beta t.$

# Newton's Method with Line Search

**given** a starting point $x \in \mathbf{dom}\, f$, tolerance $\epsilon > 0$.

**repeat**

1. *Compute the Newton step and decrement.*
   $\Delta x_{\text{nt}} := -\nabla^2 f(x)^{-1} \nabla f(x); \quad \lambda^2 := \nabla f(x)^T \nabla^2 f(x)^{-1} \nabla f(x).$
2. *Stopping criterion.* **quit** if $\lambda^2/2 \leq \epsilon$.
3. *Line search.* Choose step size $t$ by backtracking line search.
4. *Update.* $x := x + t\Delta x_{\text{nt}}$.

# Newton's Method for Strongly Convex Functions

- Strong convexity with parameters $\beta_-, \beta_+$
- Lipschitz continuity of the Hessian

$$\|\nabla^2 f(x) - \nabla^2 f(y)\|_2 \leq L\|x - y\|_2^2$$

for some constant $L > 0$

- **Basic convergence result:** The number of iterations for $\epsilon$ approximate solution in objective value is bounded by

$$T := \text{constant} \times \frac{f(x_0) - f^*}{\beta_- / \beta_+^2} + \log_2 \log_2 \left(\frac{\epsilon_0}{\epsilon}\right)$$

where $\epsilon_0 = 2\beta_-^3 / L^2$. Computational complexity: $O((nd^2 + nd)T)$

# Affine Invariance of Newton's Method

- The previous analysis can be improved
- The key insight is that Newton's Method is invariant under linear transformations
- Newton's Method for $f(x)$ is $x_{t+1} = x_t - \left(\nabla^2 f(x)\right)^{-1} \nabla f(x)$
- Consider a linear invertible transformation $y = Ax$ and $g(y) = f(A^{-1}y)$. Then Newton's Method for $g(y)$ is given by

$$
\begin{aligned}
y_{t+1} &= y_t - \left(\nabla^2 g(y_t)\right)^{-1} \nabla g(y_t) \\
&= Ax_t - (A^{-T}\nabla^2 f(x_t)A^{-1})^{-1}A^{-T}\nabla f(x) \\
&= Ax_t - A\nabla^2 f(x_t)^{-1}\nabla f(x_t) = Ax_{t+1}
\end{aligned}
$$

# Self-concordant Functions in $\mathbb{R}$

- A function $f : \mathbb{R} \to \mathbb{R}$ is self-concordant when $f$ is convex and

$$f'''(x) \leq 2f''(x)^{3/2}$$

  for all $x$ in the domain of $f$.

- Examples: linear and quadratic functions, negative logarithm

- One can use a constant $k$ other than 2 in the definition. The number $2$ is used in the definition so that $-\log(x)$ is self-concordant

# Self-concordant Functions in $\mathbb{R}^d$

- A function $f : \mathbb{R}^d \to \mathbb{R}$ is self-concordant when it is self-concordant along every line, i.e.,

(i) $f$ is convex

(ii) $g(t) := f(x + tv)$ is self-concordant for all $x$ in the domain of $f$ and all $v$

# Self-concordant Functions in $\mathbb{R}^d$

- Scaling with a positive factor of at least $1$ preserves self-concordance:

$$f \text{ is self concordant} \implies \alpha f \text{ is self concordant} \quad \text{for } \alpha \geq 1$$

- Addition preserves self-concordance

$$f_1 \text{ and } f_2 \text{ is self concordant} \implies f_1 + f_2 \text{ is self concordant}$$

- if $f(x)$ is self-concordant, affine transformations $g(x) := f(Ax + b)$ are also self-concordant

- $x^T Ax + b^T x$, $-\log(x)$ and $-\log\det(X)$ are self-concordant functions

# Newton's Method for Self-concordant Functions

- Suppose $f$ is a self-concordant function
- **Theorem**

  Newton's method with line search finds an $\epsilon$ approximate point in less than

  $$T := \text{constant} \times (f(x_0) - f^*) + \log_2 \log_2 \frac{1}{\epsilon}$$

  iterations.

- Computational complexity: $T\times$ (cost of Newton Step)
  due to Nesterov and Nemirovski

# Interior Point Programming

- Logarithmic Barrier Method

  Goal:

$$\min_x f_0(x) \text{ s.t. } f_i(x) \leq 0,\ i = 1, ..., n$$

  Indicator penalized form

$$\min_x f_0(x) + \sum_{i=1}^{n} \mathbb{I}(f_i(x))$$

  where $\mathbb{I}$ is a $\{0, \infty\}$ valued indicator function

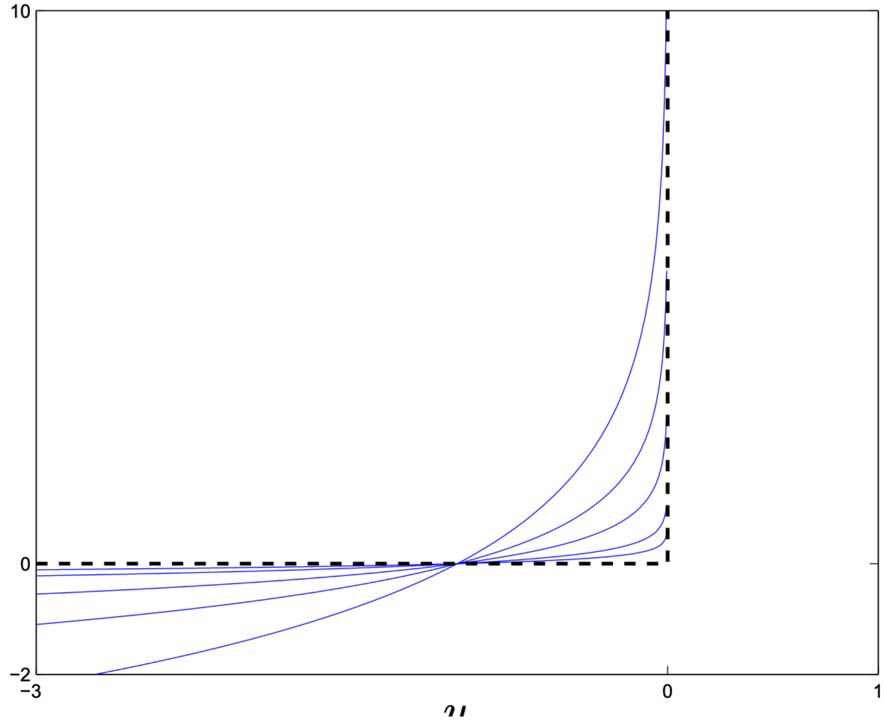
- Approximate the indicator via $\frac{-\log(-f_i(x))}{t}$

$$x^*(t) = \arg\min_x f_0(x) - \frac{1}{t}\sum_{i=1}^{n} \log(-f_i(x))$$
$$= \arg\min_x t f_0(x) - \sum_{i=1}^{n} \log(-f_i(x))$$

- $t > 0$ is the barrier parameter

- $x^*(t), t > 0$ is called the *central path*

# Interior Point Programming



# Example: Linear Programming

- LP in standard form where $A \in R^{n \times d}$

$$\min_{Ax \leq b} \; c^T x$$

- Central path

$$\arg\min_x tc^T x - \sum_{i=1}^{n} \log(b_i - a_i^T x)$$

- self-concordant function

- Hessian $\nabla^2 f(x) = A^T diag\left(\frac{1}{(b_i - a_i^T x)^2}\right) A$ takes $O(nd^2)$ operations

## Barrier Method for Constrained Convex Programs

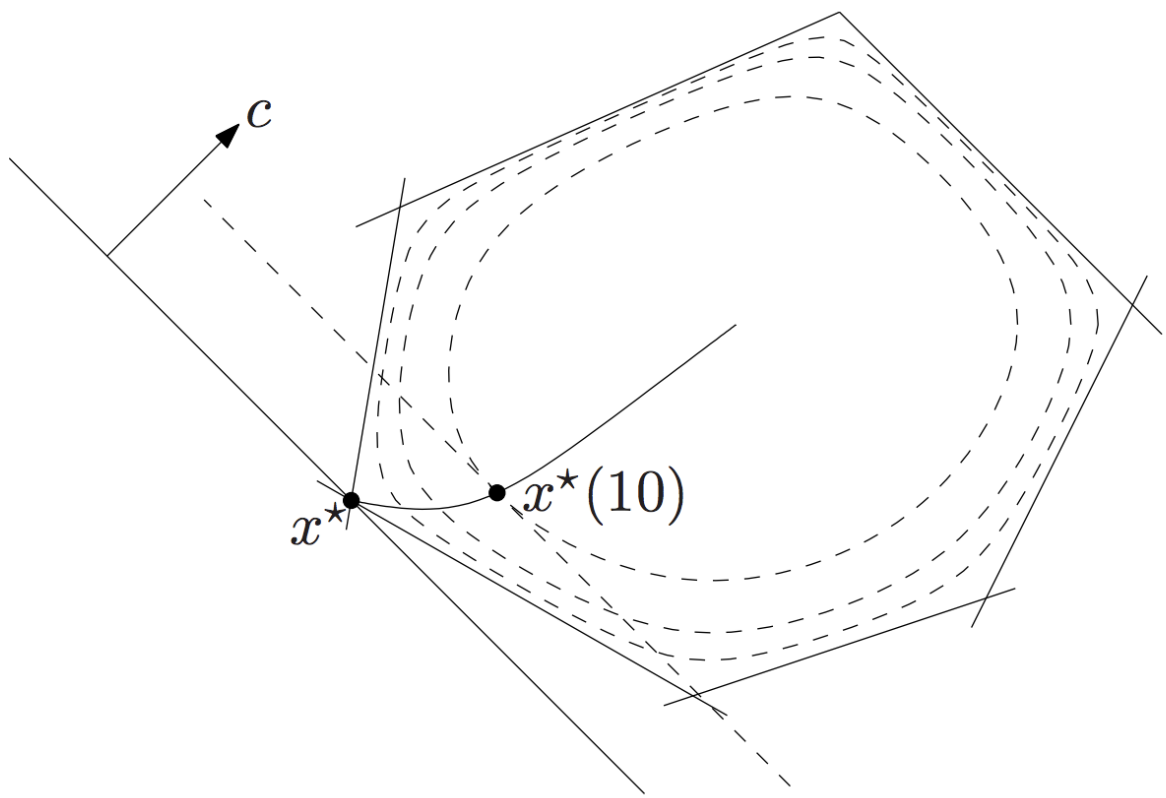
$$p^* = \min f_0(x) \text{ s.t. } f_i(x) \le 0,\ i = 1, \ldots, n$$

Suppose that $f_0, f_1, \ldots, f_n$ are twice differentiable. Define

$$x^*(t) := \min_x t f_0(x) - \sum_{i=1}^{n} \log(-f_i(x))$$

1. Centering step. Compute $x^*(t)$ via Newton's Method starting at $x$
2. Update $x := x^*(t)$
3. Stopping criterion. quit if $n/t < \epsilon$
4. Increase $t$. $t := \mu t$

# Central path for an LP



# Other Self-concordant (sc) Barrier Functions

- $-\log\det X$ is an sc barrier for the positive semidefinite cone
- $-\log(x^T Ax + b^T x + c)$ is an sc barrier for the convex set $x^T Ax + b^T x + c > 0$ when $A \succeq 0$
- $-\log(y^2 - x^x)$ is an sc barrier for the second order cone $\|x\|_2 \leq y$

# Barrier Method for Constrained Convex Programs

- terminates with $f_0(x^*(t)) - p^* \leq \epsilon$
- choice of $\mu$ involves a trade-off: large $\mu$ means fewer outer iterations, more inner (Newton) iterations. Typical values of $\mu = 10 - 20$

# Optimality gap of the central path

- Central path $x^*(t) = \arg\min_x tf_0(x) - \sum_{i=1}^n \log(-f_i(x))$

- Optimality conditions $x^*(t)$ (necessary and sufficient)

$$t\nabla f_0(x^*) + \sum_{i=1}^n \frac{1}{-f_i(x^*(t))}\nabla f_i(x^*(t)) = 0$$

- $x^*(t)$ minimizes the Lagrangian for the original problem for $\lambda = -\frac{1}{tf_i(x^*(t))}$

$$\nabla_x L(x, \lambda) = \nabla f_0(x) + \sum_{i=1}^n \lambda_i \nabla f_i(x) = 0$$

- $\lambda^*(t) = -\frac{1}{tf_i(x^*(t))} > 0$ is dual feasible and provides a lower-bound

$$\begin{aligned}
\min_{x \text{ s.t. } f_i(x) \le 0 \forall i} f_0(x) &\ge \max_{\lambda \succeq 0} \min_x f_0(x) + \sum_{i=1}^n \lambda_i f_i(x) \\
&\ge \min_x f_0(x) + \sum_{i=1}^n \lambda^*(t) f_i(x) \\
&= f_0(x^*(t)) + \sum_{i=1}^n \lambda^*(t) f_i(x^*(t)) \\
&= f_0(x^*(t)) - \sum_{i=1}^n \frac{f_i(x^*(t))}{tf_i(x^*(t))} = f_0(x^*(t)) - \frac{n}{t}
\end{aligned}$$

Therefore optimality gap is at most $n/t$

## Complexity Analysis: Number of centering steps

- Assuming that we can find $x^*(t) = \arg\min_x t f_0(x) - \sum_{i=1}^{n} \log(-f_i(x))$ via Newton's method for $t = t^0, \mu t^0, \mu^2 t^0, \ldots$, the optimality gap after $k$ centering steps is $\frac{n}{\mu^k t^0}$

- Accuracy $\epsilon$ is achieved after

$$\frac{\log(m/(\epsilon t^0))}{\log \mu}$$

centering steps, plus the initial centering step

# Complexity Analysis: Number of Newton Iterations

- Number of Newton iterations per centering step is bounded by

$$T := \text{constant} \times (f(x_0) - \min_x f(x)) + \log_2 \log_2 \frac{1}{\epsilon}$$

- Bound on the effort of computing $x^*(\mu t)$ starting at $x = x^*(t)$ depends on the initial optimality gap $f(x_0) - \min f(x)$ where $f(x) := t f_0(x) + \sum_{i=1}^n \log(-f_i(x))$

- it can be shown that (see Chapter 11.5 in Convex Optimization)

$$T \leq \text{constant} \times \frac{n(\mu - 1 - \log \mu)}{\gamma} + \log_2 \log_2 \frac{1}{\epsilon}$$

- number of outer (centering) iterations is $\frac{\log(n/(\epsilon t^{(0)}))}{\log \mu}$

- total number of Newton iterations $N := \frac{\log(n/(\epsilon t^{(0)}))}{\log \mu} \frac{n(\mu - 1 - \log \mu)}{\gamma}$

- confirms the trade-off in the choice of $\mu$

- for $\mu = 1 + 1/\sqrt{n}$, total number of Newton iterations
  $N = O(\sqrt{n} \log\left(\frac{n/t^{(0)}}{\epsilon}\right))$

- this proves the polynomial-time complexity of barrier method for convex programming

- this choice of $\mu$ optimizes worst-case complexity. In practice we choose $\mu$ fixed, e.g., $\mu = 10, \ldots, 20$. The number of outer iterations is in the tens and not very sensitive for $\mu \geq 10$.

# Numerical Example

- We solve a Second Order Cone Program

$$\begin{aligned} &\min f^T x \\ &\text{s.t. } \|A_i x + b_i\|_2 \le c_i^T x + d_i,\ i = 1, \ldots, n \end{aligned}$$

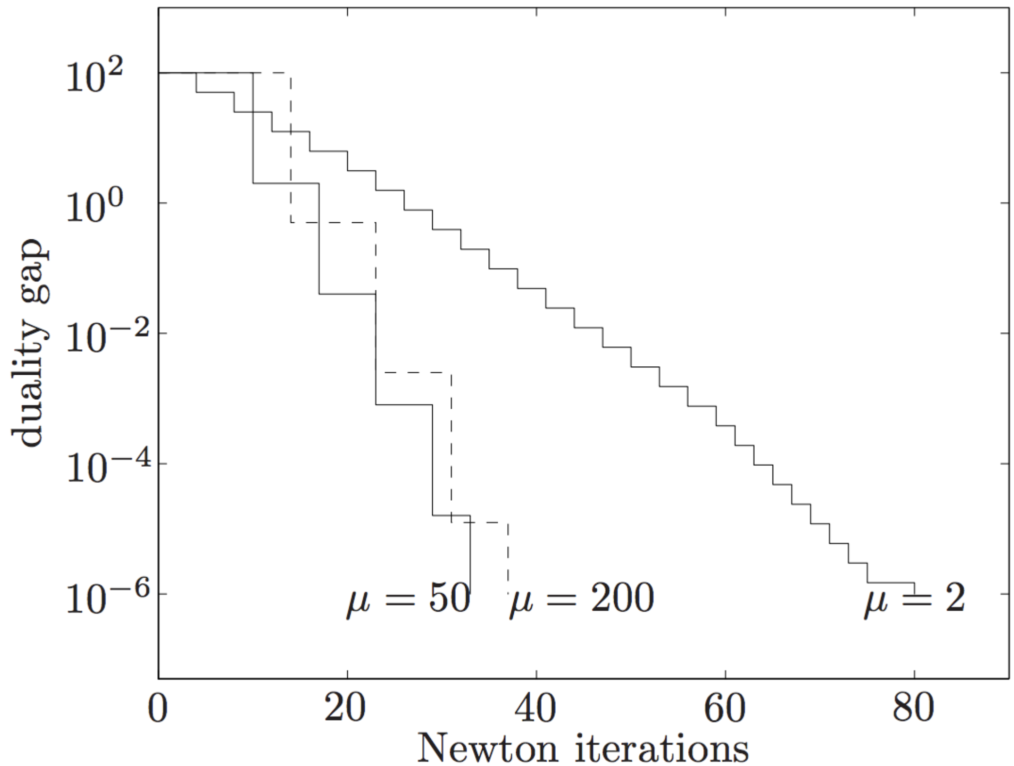
using the sc barrier $-\sum_{i=1}^{n} \log\left((c_i^T x + d_i)^2 - \|A_i x + b\|_2^2\right)$

- The central path is given by

$$x^*(t) = \arg\min_x t f^T x - \sum_{i=1}^{n} \log\left((c_i^T x + d_i)^2 - \|A_i x + b\|_2^2\right)$$

# Numerical Example

- Randomly generated problem instances where $n = 50$ and $x \in \mathbb{R}^{50}$



# Reformulating Non-differentiable Objectives

- Example: Robust regression

$$\min_x \|Ax - b\|_1$$

- Reformulation

$$\begin{array}{llcll} \min_{x,y} & \|y\|_1 & = & \min_{x,y,s} & 1^T s \\ \text{s.t.} & Ax - b = y & & \text{s.t.} & -s_i \le y_i \le s_i \, \forall i \\ & & & & Ax - b = y \end{array}$$

# Conclusions

- Interior Point (barrier) methods run in provably polynomial-time for convex optimization when we have self-concordant barriers
- They are also very efficient in practice
- Main computational load is solving 20-30 linear systems for the Newton iterations
- There are also primal-dual interior methods which are more efficient