Subgradients

- subgradients
- strong and weak subgradient calculus
- optimality conditions via subgradients
- directional derivatives
- generalized subdifferential for non-convex functions

Basic inequality

recall basic inequality for convex differentiable f:

$$f(y) \geq f(x) + \nabla f(x)^T (y - x)$$

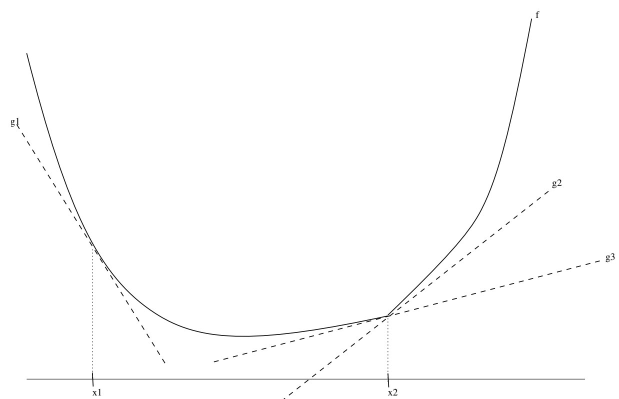
- first-order approximation of f at x is global underestimator
- $(\nabla f(x), -1)$ supports $\mathbf{epi}\, f$ at $(x, f(x))$

what if f is not differentiable?

Subgradient of a function

g is a **subgradient** of f (not necessarily convex) at x if

$$f(y) \geq f(x) + g^T(y - x) \quad \text{for all } y$$



g_2, g_3 are subgradients at x_2; g_1 is a subgradient at x_1

- g is a subgradient of f at x iff $(g, -1)$ supports $\mathbf{epi}\, f$ at $(x, f(x))$
- g is a subgradient iff $f(x) + g^T(y - x)$ is a global (affine) underestimator of f
- if f is convex and differentiable, $\nabla f(x)$ is a subgradient of f at x

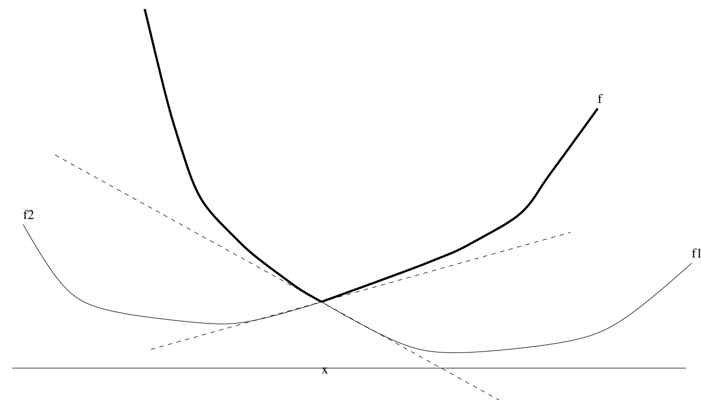
subgradients come up in several contexts:

- algorithms for nondifferentiable convex optimization
- convex analysis, $e.g.$, optimality conditions, duality for nondifferentiable problems

(if $f(y) \leq f(x) + g^T(y - x)$ for all y, then g is a **supergradient**)

Example

$f = \max\{f_1, f_2\}$, with f_1, f_2 convex and differentiable



- $f_1(x_0) > f_2(x_0)$: unique subgradient $g = \nabla f_1(x_0)$
- $f_2(x_0) > f_1(x_0)$: unique subgradient $g = \nabla f_2(x_0)$
- $f_1(x_0) = f_2(x_0)$: subgradients form a line segment $[\nabla f_1(x_0), \nabla f_2(x_0)]$

Subdifferential

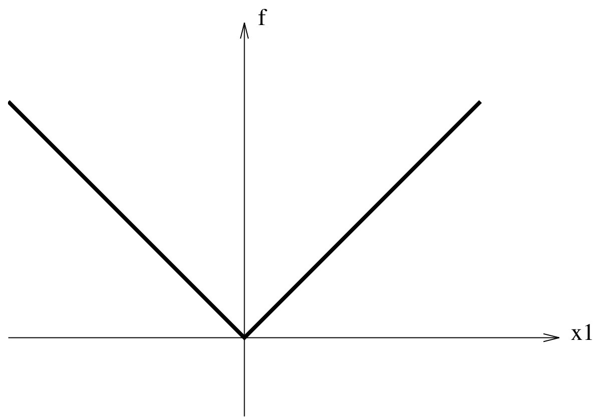
- set of all subgradients of f at x is called the **subdifferential** of f at x, denoted $\partial f(x)$
- $\partial f(x)$ is a closed convex set (can be empty)

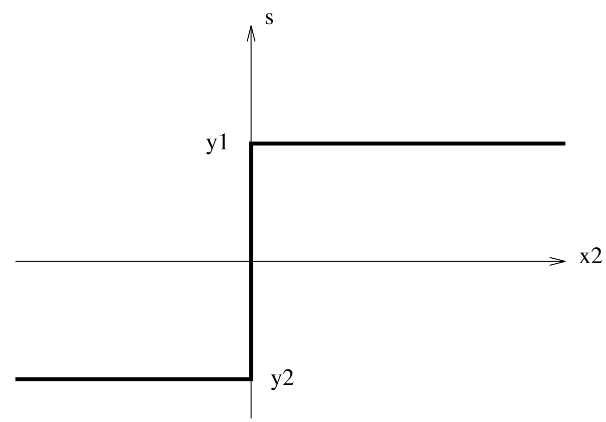
if f is convex,

- $\partial f(x)$ is nonempty, for $x \in \mathbf{relint\,dom}\, f$
- $\partial f(x) = \{\nabla f(x)\}$, if f is differentiable at x
- if $\partial f(x) = \{g\}$, then f is differentiable at x and $g = \nabla f(x)$

Example

$f(x) = |x|$





righthand plot shows $\bigcup \{(x, g) \mid x \in \mathbf{R},\ g \in \partial f(x)\}$

Subgradient calculus

- **weak subgradient calculus**: formulas for finding *one* subgradient $g \in \partial f(x)$
- **strong subgradient calculus**: formulas for finding the whole subdifferential $\partial f(x)$, $i.e.$, *all* subgradients of f at x
- many algorithms for nondifferentiable convex optimization require only *one* subgradient at each step, so weak calculus suffices
- some algorithms, optimality conditions, etc., need whole subdifferential
- roughly speaking: if you can compute $f(x)$, you can usually compute a $g \in \partial f(x)$
- we'll assume that f is convex, and $x \in \mathbf{relint}\,\mathbf{dom}\, f$

Some basic rules

- $\partial f(x) = \{\nabla f(x)\}$ if f is differentiable at x
- **scaling:** $\partial(\alpha f) = \alpha \partial f$ (if $\alpha > 0$)
- **addition:** $\partial(f_1 + f_2) = \partial f_1 + \partial f_2$ (RHS is addition of point-to-set mappings)
- **affine transformation of variables:** if $g(x) = f(Ax + b)$, then $\partial g(x) = A^T \partial f(Ax + b)$
- **finite pointwise maximum:** if $f = \max_{i=1,\ldots,m} f_i$, then

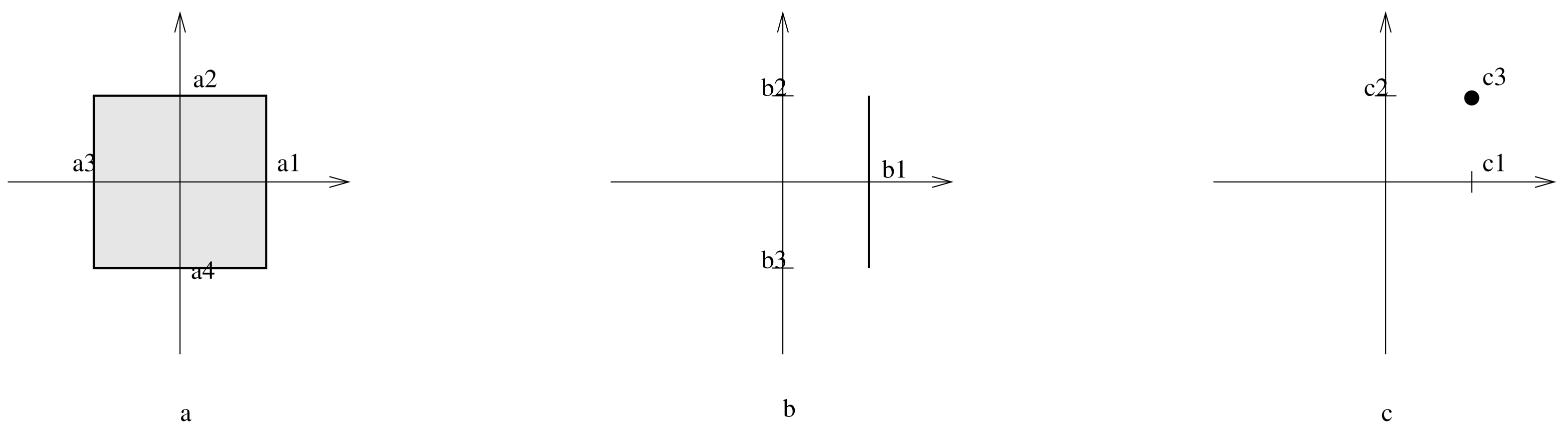
$$\partial f(x) = \mathbf{Co} \bigcup \{\partial f_i(x) \mid f_i(x) = f(x)\},$$

$i.e.$, convex hull of union of subdifferentials of 'active' functions at x

$f(x) = \max\{f_1(x), \ldots, f_m(x)\}$, with $f_1, \ldots, f_m$ differentiable

$$\partial f(x) = \mathbf{Co}\{\nabla f_i(x) \mid f_i(x) = f(x)\}$$

example: $f(x) = \|x\|_1 = \max\{s^T x \mid s_i \in \{-1, 1\}\}$



Pointwise supremum

if $f = \sup_{\alpha \in \mathcal{A}} f_\alpha$,

$$\mathbf{cl}\,\mathbf{Co} \bigcup \{\partial f_\beta(x) \mid f_\beta(x) = f(x)\} \subseteq \partial f(x)$$

(usually get equality, but requires some technical conditions to hold, $e.g.$, $\mathcal{A}$ compact, f_α cts in x and α)

roughly speaking, $\partial f(x)$ is closure of convex hull of union of subdifferentials of active functions

Weak rule for pointwise supremum

$$f = \sup_{\alpha \in \mathcal{A}} f_\alpha$$

- find *any* β for which $f_\beta(x) = f(x)$ (assuming supremum is achieved)
- choose *any* $g \in \partial f_\beta(x)$
- then, $g \in \partial f(x)$

example

$$f(x) = \lambda_{\max}(A(x)) = \sup_{\|y\|_2=1} y^T A(x) y$$

where $A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n$, $A_i \in \mathbf{S}^k$

- f is pointwise supremum of $g_y(x) = y^T A(x) y$ over $\|y\|_2 = 1$
- g_y is affine in x, with $\nabla g_y(x) = (y^T A_1 y, \ldots, y^T A_n y)$
- hence, $\partial f(x) \supseteq \mathbf{Co}\,\{\nabla g_y \mid A(x)y = \lambda_{\max}(A(x))y,\ \|y\|_2 = 1\}$ (in fact equality holds here)

to find **one** subgradient at x, can choose **any** unit eigenvector y associated with $\lambda_{\max}(A(x))$; then

$$(y^T A_1 y, \ldots, y^T A_n y) \in \partial f(x)$$

Expectation

- $f(x) = \mathbf{E}\, f(x, \omega)$, with f convex in x for each ω, ω a random variable
- for each ω, choose *any* $g_\omega \in \partial_f(x, \omega)$ (so $\omega \mapsto g_\omega$ is a function)
- then, $g = \mathbf{E}\, g_\omega \in \partial f(x)$

Monte Carlo method for (approximately) computing $f(x)$ and *a* $g \in \partial f(x)$:

- generate independent samples $\omega_1, \ldots, \omega_K$ from distribution of ω
- $f(x) \approx (1/K) \sum_{i=1}^K f(x, \omega_i)$
- for each i choose $g_i \in \partial_x f(x, \omega_i)$
- $g = (1/K) \sum_{i=1}^K g_i$ is an (approximate) subgradient
  (more on this later)

Minimization

define $g(y)$ as the optimal value of

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq y_i, \quad i = 1, \ldots, m \end{array}$$

(f_i convex; variable x)

with $\lambda^\star$ an optimal dual variable, we have

$$g(z) \geq g(y) - \sum_{i=1}^{m} \lambda_i^\star (z_i - y_i)$$

$i.e.$, $-\lambda^\star$ is a subgradient of g at y

Composition

- $f(x) = h(f_1(x), \ldots, f_k(x))$, with h convex nondecreasing, f_i convex
- find $q \in \partial h(f_1(x), \ldots, f_k(x))$, $g_i \in \partial f_i(x)$
- then, $g = q_1 g_1 + \cdots + q_k g_k \in \partial f(x)$
- reduces to standard formula for differentiable h, f_i

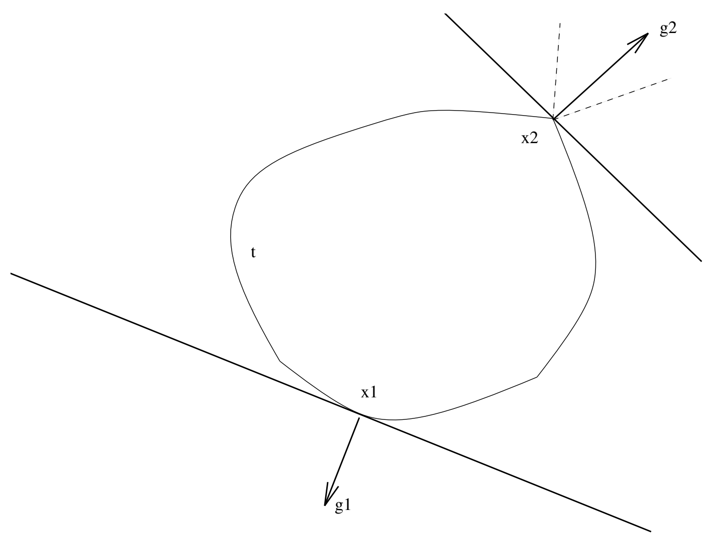
proof:

$$\begin{aligned} f(y) &= h(f_1(y), \ldots, f_k(y)) \\ &\geq h(f_1(x) + g_1^T(y-x), \ldots, f_k(x) + g_k^T(y-x)) \\ &\geq h(f_1(x), \ldots, f_k(x)) + q^T(g_1^T(y-x), \ldots, g_k^T(y-x)) \\ &= f(x) + g^T(y-x) \end{aligned}$$

Subgradients and sublevel sets

g is a subgradient at x means $f(y) \geq f(x) + g^T(y - x)$

hence $f(y) \leq f(x) \Longrightarrow g^T(y - x) \leq 0$



- f differentiable at x_0: $\nabla f(x_0)$ is normal to the sublevel set $\{x \mid f(x) \leq f(x_0)\}$

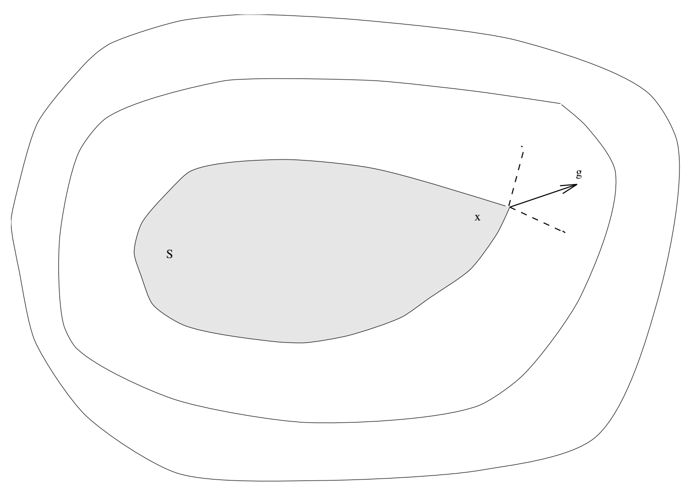
- f nondifferentiable at x_0: subgradient defines a supporting hyperplane to sublevel set through x_0

Quasigradients

$g \neq 0$ is a **quasigradient** of f at x if

$$g^T(y - x) \geq 0 \implies f(y) \geq f(x)$$

holds for all y



quasigradients at x form a cone

example:

$$f(x) = \frac{a^T x + b}{c^T x + d}, \quad (\mathbf{dom}\, f = \{x \mid c^T x + d > 0\})$$

$g = a - f(x_0)c$ is a quasigradient at x_0

proof: for $c^T x + d > 0$:

$$a^T(x - x_0) \geq f(x_0)c^T(x - x_0) \Longrightarrow f(x) \geq f(x_0)$$

example: degree of $a_1 + a_2 t + \cdots + a_n t^{n-1}$

$$f(a) = \min\{i \mid a_{i+2} = \cdots = a_n = 0\}$$

$g = \mathsf{sign}(a_{k+1}) e_{k+1}$ (with $k = f(a)$) is a quasigradient at $a \neq 0$

proof:

$$g^T(b - a) = \mathsf{sign}(a_{k+1}) b_{k+1} - |a_{k+1}| \geq 0$$

implies $b_{k+1} \neq 0$

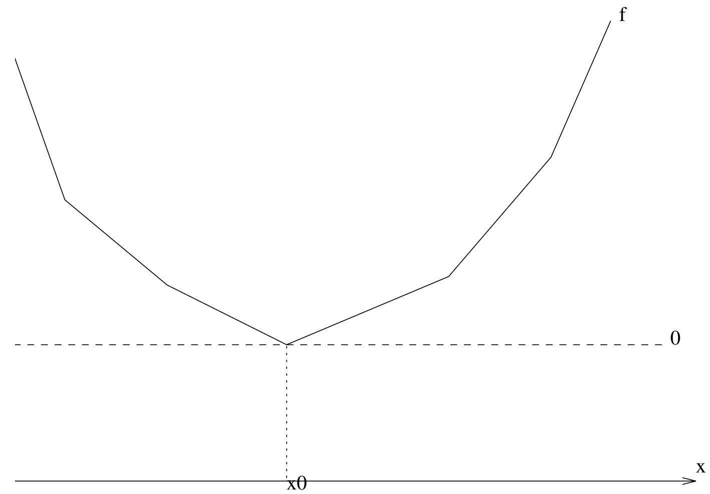
Optimality conditions — unconstrained

recall for f convex, differentiable,

$$f(x^\star) = \inf_x f(x) \iff 0 = \nabla f(x^\star)$$

generalization to nondifferentiable convex f:

$$f(x^\star) = \inf_x f(x) \iff 0 \in \partial f(x^\star)$$



proof. by definition (!)

$$f(y) \geq f(x^\star) + 0^T(y - x^\star) \text{ for all } y \iff 0 \in \partial f(x^\star)$$

. . . seems trivial but isn't

Example: piecewise linear minimization

$f(x) = \max_{i=1,\ldots,m}(a_i^T x + b_i)$

$x^\star$ minimizes $f \iff 0 \in \partial f(x^\star) = \mathbf{Co}\{a_i \mid a_i^T x^\star + b_i = f(x^\star)\}$

$\iff$ there is a λ with

$$\lambda \succeq 0, \qquad \mathbf{1}^T\lambda = 1, \qquad \sum_{i=1}^m \lambda_i a_i = 0$$

where $\lambda_i = 0$ if $a_i^T x^\star + b_i < f(x^\star)$

. . . but these are the KKT conditions for the epigraph form

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & a_i^T x + b_i \leq t, \quad i = 1, \ldots, m \end{array}$$

with dual

$$\begin{array}{ll} \text{maximize} & b^T \lambda \\ \text{subject to} & \lambda \succeq 0, \qquad A^T \lambda = 0, \qquad \mathbf{1}^T \lambda = 1 \end{array}$$

Optimality conditions — constrained

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \; i = 1, \ldots, m \end{array}$$

we assume

- f_i convex, defined on $\mathbf{R}^n$ (hence subdifferentiable)
- strict feasibility (Slater's condition)

$x^\star$ is primal optimal ($\lambda^\star$ is dual optimal) iff

$$\begin{aligned} & f_i(x^\star) \leq 0, \quad \lambda_i^\star \geq 0 \\ & 0 \in \partial f_0(x^\star) + \textstyle\sum_{i=1}^m \lambda_i^\star \partial f_i(x^\star) \\ & \lambda_i^\star f_i(x^\star) = 0 \end{aligned}$$

. . . generalizes KKT for nondifferentiable f_i

Directional derivative

directional derivative of f at x in the direction δx is

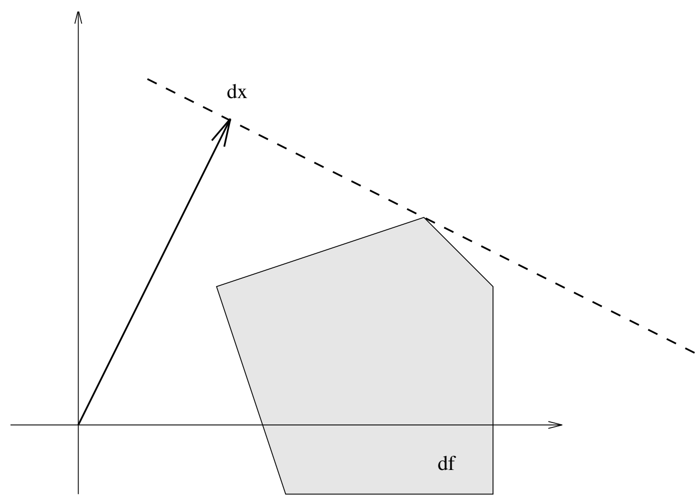
$$f'(x;\delta x) \stackrel{\Delta}{=} \lim_{h \searrow 0} \frac{f(x+h\delta x)-f(x)}{h}$$

can be $+\infty$ or $-\infty$

- f convex, finite near $x \implies f'(x;\delta x)$ exists
- f differentiable at x if and only if, for some g $(= \nabla f(x))$ and all δx, $f'(x;\delta x) = g^T\delta x$ (*i.e.*, $f'(x;\delta x)$ is a linear function of δx)

Directional derivative and subdifferential

general formula for convex f: $f'(x;\delta x) = \sup_{g \in \partial f(x)} g^T \delta x$



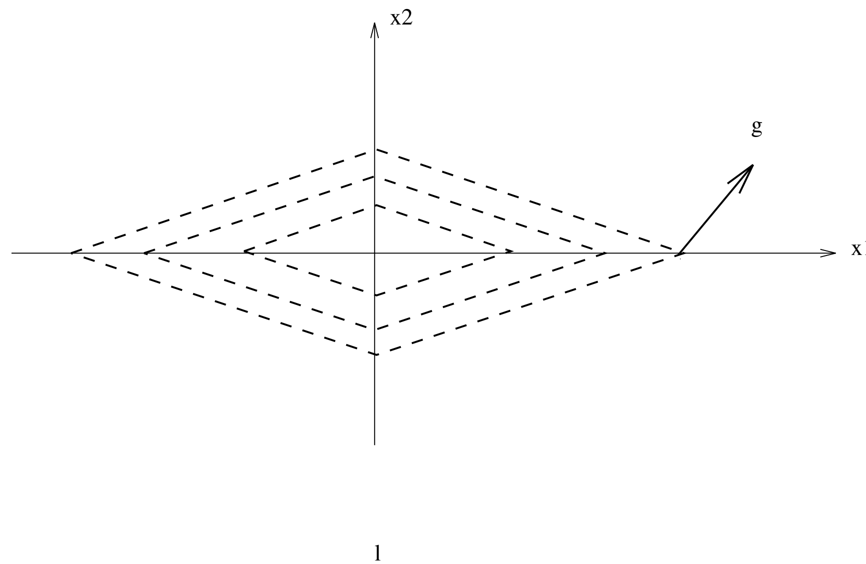
Descent directions

δx is a **descent direction** for f at x if $f'(x; \delta x) < 0$

for differentiable f, $\delta x = -\nabla f(x)$ is always a descent direction (except when it is zero)

warning: for nondifferentiable (convex) functions, $\delta x = -g$, with $g \in \partial f(x)$, need not be descent direction

example: $f(x) = |x_1| + 2|x_2|$



Subgradients and distance to sublevel sets

if f is convex, $f(z) < f(x)$, $g \in \partial f(x)$, then for small $t > 0$,

$$\|x - tg - z\|_2 < \|x - z\|_2$$

thus $-g$ is descent direction for $\|x - z\|_2$, for **any** z with $f(z) < f(x)$ (*e.g.*, $x^\star$)

negative subgradient is descent direction for distance to optimal point

proof: $$\begin{aligned} \|x - tg - z\|_2^2 &= \|x - z\|_2^2 - 2tg^T(x - z) + t^2\|g\|_2^2 \\ &\leq \|x - z\|_2^2 - 2t(f(x) - f(z)) + t^2\|g\|_2^2 \end{aligned}$$

Descent directions and optimality

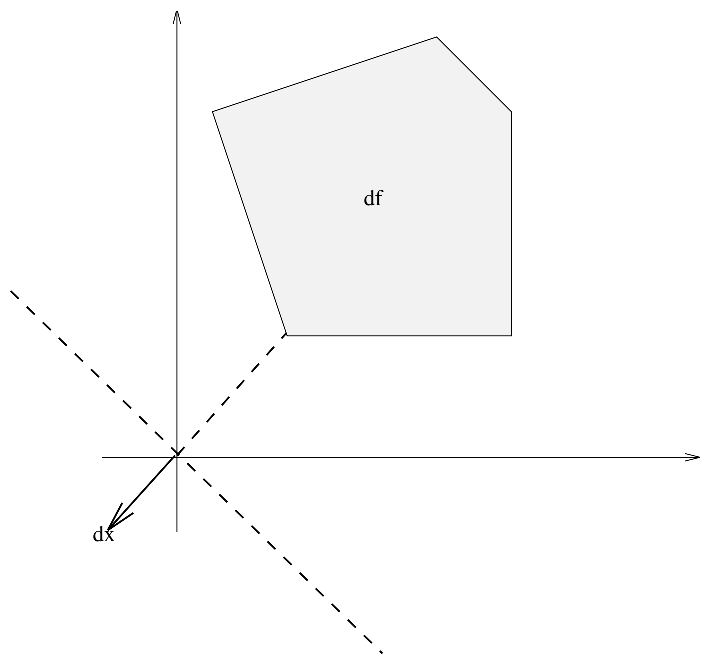
fact: for f convex, finite near x, either

- $0 \in \partial f(x)$ (in which case x minimizes f), or
- there is a descent direction for f at x

i.e., x is optimal (minimizes f) iff there is no descent direction for f at x

proof: define $\delta x_{\mathrm{sd}} = -\operatorname*{argmin}_{z \in \partial f(x)} \|z\|_2$

if $\delta x_{\mathrm{sd}} = 0$, then $0 \in \partial f(x)$, so x is optimal; otherwise
$f'(x; \delta x_{\mathrm{sd}}) = -\left(\inf_{z \in \partial f(x)} \|z\|_2\right)^2 < 0$, so δx_{sd} is a descent direction



idea extends to constrained case (feasible descent direction)

Non-convex and non-smooth functions

Clarke subdifferential of f at x is

$$\partial_C f(x) = \mathbf{Co}\left\{\lim_{k\to\infty} \nabla f(x_k) \mid x_k \to x,\ \nabla f(x_k) \text{ exists}\right\}$$

- coincides with the ordinary subdifferential $\partial f(x)$ when f is convex

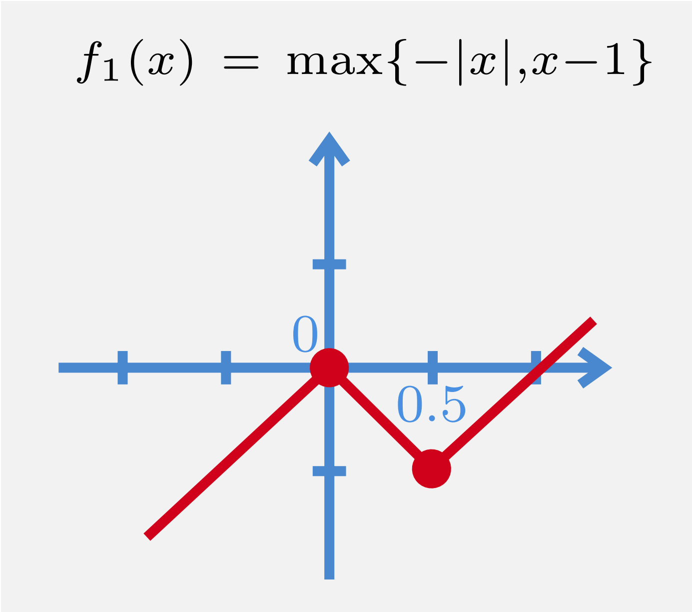
Local minima and maxima

$$\text{minimize } f(x)$$

$$x \text{ is a local minimum or maximum of } f(x) \quad \Longrightarrow \quad 0 \in \partial_C f(x).$$

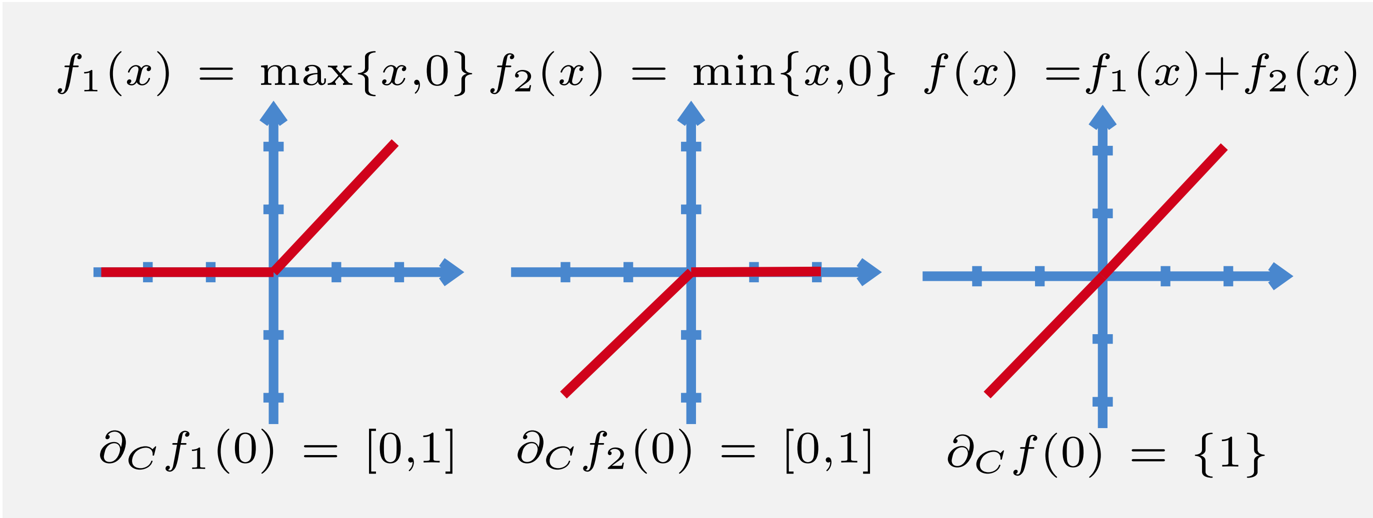
- $f(x)$ is assumed to be locally Lipschitz, non-convex and non-differentiable
- the reverse implication does not hold in general
- can be extended to constrained non-convex optimization

Example



- $x = 0$ is a local maximum and $x = \frac{1}{2}$ is a local minimum
- $0 \in \partial_C f(0) = [-1, 1]$ and $0 \in \partial_C f(\frac{1}{2}) = [-1, 1]$

Clarke subdifferential of a sum



- weak sum rule holds: $\nabla_C(f_1 + f_2) \subseteq \nabla_C f_1 + \nabla_C f_2$
- equality holds when functions are **subdifferentially regular** (see lecture notes for the definition)

References

References

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